On the Design of Spatial $4C$ Mechanisms for Rigid-Body Guidance Through 4 Positions

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Abstract

This paper presents a procedure for determining the fixed and moving congruences associated with four finitely separated spatial positions. Furthermore, a methodology is derived for selecting the lines from the congruences which define the joint axes of a $4C$ mechanism. The result is a design procedure for performing the kinematic dimensional synthesis of spatial $4C$ mechanisms for four position rigid body guidance.

Associated with four finitely separated positions in space are a fixed and a moving congruence. These congruences are a two dimensional set of lines, where each line defines the axis of a cylindrical joint that guides a body through the four prescribed positions. In order to uniquely determine a $4C$ mechanism from the congruences four free parameters must be specified. We present procedures for determining these free design parameters which result in mechanisms with joint axes that are nearest to some desired location. Moreover, included is a detailed numerical example illustrating the design process.

1 Introduction

In this paper we present the spatial generalization of the center point and circle point curves of planar kinematics and the center axis and circle axis cones of spherical kinematics called the fixed and moving congruences. The congruences are the set of lines that define the axes of CC dyads that guide a body through four prescribed positions in space. A compatible pair of fixed and moving axes maintains a constant normal distance and angle in each of the four positions of the moving body. The fixed and moving axes define the C joints of the dyad. A C joint is a two degree of freedom cylindrical joint which allows translation along and rotation about a line in space.
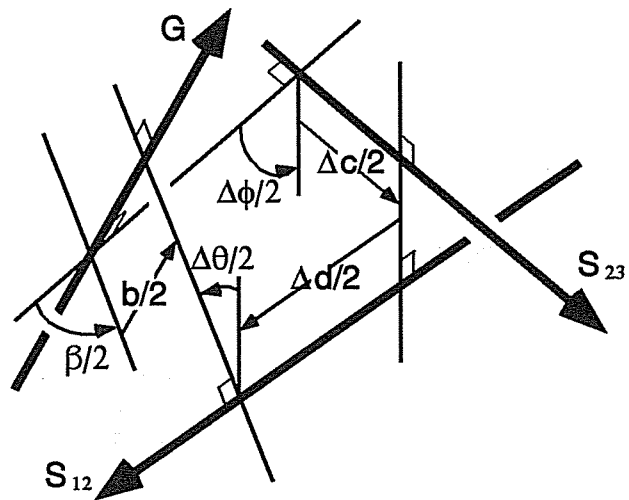

Figure 1: The Spatial Triangle

Two compatible CC dyads are combined to form a spatial closed kinematic chain, a spatial $4C$ mechanism, see Fig. 2.

The congruences of lines are a two dimensional set of lines. We compute the congruences by employing the spatial triangle technique presented by Murray and McCarthy (1994). The result is a parameterized set of lines. Our goal is to derive techniques for numerically determining the congruences and for selecting lines from them to define compatible CC dyads. We hope that these techniques will in turn facilitate the creation of computer-aided design software for spatial $4C$ mechanism design.

2 The Spatial Triangle

Before proceeding to the numerical generation of the fixed and moving congruences it is instructive to re-
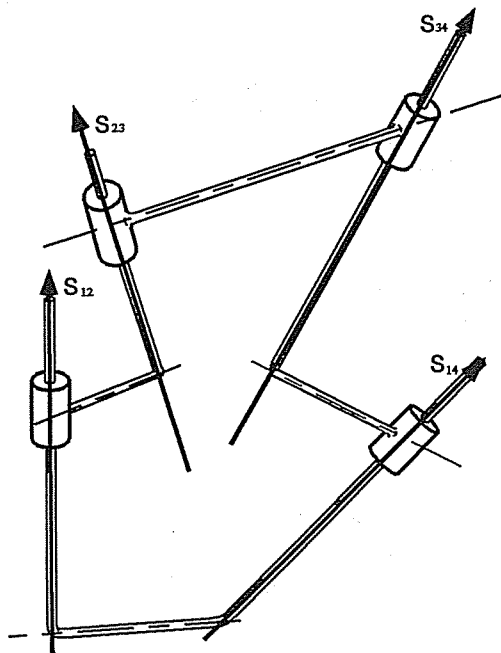
Figure 2: The Parameterizing Spatial $4C$ Mechanism

view the spatial triangle technique of Murray and McCarthy (1994) for synthesizing CC dyads compatible with four spatial positions. In Murray and McCarthy (1994) it is shown that the spatial triangle prescribed by the relative screw axes S_{23} and S_{12} with internal dual angles $\frac{\Delta\hat{\phi}}{2}$ and $\frac{\Delta\hat{\theta}}{2}$ defines the coordinates of a fixed line G of a CC dyad compatible with four spatial positions, see Fig. 1. The dual vector equation of the spatial triangle is written as,

$$
\begin{aligned}
\sin\frac{\hat{\beta}}{2}G = \ & \sin\frac{\Delta\hat{\theta}}{2}\cos\frac{\Delta\hat{\phi}}{2}S_{12} + \\
& \cos\frac{\Delta\hat{\theta}}{2}\sin\frac{\Delta\hat{\phi}}{2}S_{23} + \\
& \sin\frac{\Delta\hat{\theta}}{2}\sin\frac{\Delta\hat{\phi}}{2}S_{12}\times S_{23}
\end{aligned}
\tag{1}
$$

$$
\cos\frac{\hat{\beta}}{2} = \cos\frac{\Delta\hat{\theta}}{2}\cos\frac{\Delta\hat{\phi}}{2} + \\
\sin\frac{\Delta\hat{\theta}}{2}\sin\frac{\Delta\hat{\phi}}{2}S_{12}\cdot S_{23}
\tag{2}
$$

In order to solve Eq. 1 and Eq. 2 for the desired line G the relationships between the spatial triangle, the complementary screw quadrilateral, and the $4C$ mechanism corresponding to the complementary screw quadrilateral must be maintained. We review

those relationships here and outline the procedure for determining the line G given four spatial positions.

The generalization of Burmester's planar four position theory to four spatial displacements leads us to consider the complementary screw quadrilateral $S_{12}S_{23}S_{34}S_{14}$, where S_{12}, S_{13}, S_{14}, S_{23}, S_{24}, and S_{34} are the six relative screw axes associated with the four prescribed spatial positions of a moving body, see Roth (1967b,1967c) and Bottema and Roth (1979). McCarthy (1993a) shows that by using the lines which define the complementary screw quadrilateral to define the axes of a spatial $4C$ mechanism one may obtain the fixed axis congruence in a parameterized form, see Fig. 2. A planar version of this result, which yields a parameterized form of the center point curve for four planar positions is found in McCarthy (1993b). The procedure presented in McCarthy (1993a) involves using the lines which define the complementary screw quadrilateral to define a spatial $4C$ mechanism and identifying the quadrilateral as the *home* configuration of the parameterizing $4C$ mechanism. This results in a $4C$ mechanism with input crank defined by the lines $S_{12}S_{23}$, fixed link defined by the lines $S_{12}S_{14}$ and coupler defined by the lines $S_{23}S_{34}$. We define the input angle $\hat{\theta}_0$ as the dual angle of the input crank in the home configuration and similarly define the coupler angle $\hat{\phi}_0$ as the dual angle between the coupler and the input crank in the home configuration. Murray and McCarthy (1994) prove that the screw axis of the displacement of the coupler of the parameterizing $4C$ mechanism, from its home configuration to any other valid assembly, is a fixed axis compatible with the given four general spatial positions. Therefore, we obtain fixed axes that are parameterized by the input angle $\hat{\theta}$ of the parameterizing linkage.

In their paper Murray and McCarthy (1994) show that solving the spatial triangle associated with the two lines which define the input crank in its home configuration, Eq. 1 and Eq. 2, results in the screw axis of the displacement of the coupler of the parameterizing $4C$ mechanism, where $\Delta\hat{\theta} = \hat{\theta} - \hat{\theta}_0$ and $\Delta\hat{\phi} = \hat{\phi} - \hat{\phi}_0$. Therefore, by solving the spatial triangle we obtain a fixed axis compatible with the given four general spatial positions which is parameterized by the input angle of the parameterizing $4C$ linkage. Note that the internal angles of the spatial triangle are given in terms of the relative input and coupler angles of the parameterizing $4C$ linkage with respect to its home configuration.

Having reviewed the relationships between the relative screw axes, the complementary screw quadri-

lateral, the parameterizing $4C$ mechanism, and the spatial triangle, we now outline the procedure for determining the fixed line G given four spatial positions.

- *From the four specified positions determine: the four relative screw axes($S_{12}, S_{23}, S_{34}, S_{14}$), the cranks lengths of the corresponding parameterizing $4C$ linkage($\hat{\alpha} = S_{12} \cdot S_{23}$, $\hat{\eta} = S_{23} \cdot S_{34}$, $\hat{\beta} = S_{14} \cdot S_{34}$, $\hat{\gamma} = S_{12} \cdot S_{14}$), and the angles $\hat{\theta}_0$ and $\hat{\phi}_0$.*

- *Select the parameter value $\hat{\theta}$ and compute the corresponding $\hat{\phi}$ by performing a kinematic analysis of the parameterizing $4C$ linkage, see Murray and McCarthy (1994), McCarthy (1993a), and Duffy (1980). [1]*

- *Compute the internal angles of the spatial triangle, $\Delta\hat{\theta}$ and $\Delta\hat{\phi}$, and solve the two dual vector triangle equations, Eq. 1 and Eq. 2, for the two unknowns G and $\hat{\beta}$.*

3 The Fixed Congruence

In the previous section we reviewed a procedure presented by Murray and McCarthy (1994) for generating fixed axes of CC dyads that guide a body through four spatial positions. We now present a method of obtaining a numerical representation of the fixed congruence which is parameterized by the input angle of the parameterizing $4C$ linkage using the spatial triangle. Recall that the fixed congruence is a two dimensional set of lines that define the fixed axes that are compatible with four spatial positions and that the solution of the spatial triangle presented by Murray and McCarthy (1994) yields one line of the fixed congruence.

Bottema and Roth (1979) and Roth (1967a) have shown that the direction of each line G determines a plane and that all of the lines in that plane that are parallel to G are members of the fixed congruence. Hence, each line G defines a unique direction and corresponding to this direction there is an infinite set of compatible screw axes. We proceed with a method for using the spatial triangle to determine another line of the congruence, G_2, which is parallel to $G_1 = G$. These two lines then define the plane associated with G_1.

By examining Eq. 1 we see that the direction of G_1 is independent of the translation along, and the

[1]In general for each θ there are two solutions for ϕ, simply let θ vary from 0 to 720 and use the first solution for $0 \leq \theta < 360$ and the second solution for $360 \leq \theta < 720$.

location of, the axes of the parameterizing $4C$ mechanism. In other words, the direction of G_1 is solely dependent upon the directions of the axes of the parameterizing linkage, this result was first presented by Roth (1967a). Therefore, to obtain G_2 with the same direction as G_1 we maintain θ and vary our choice of d, where d is the translation of the input crank of the parameterizing linkage along S_{12}, $(\hat{\theta} = \theta + \epsilon d)$, and solve Eq. 1 and Eq. 2. Hence, for a given choice of parameter θ we select two different values of d which yield two lines G_1 and G_2. These two lines then define a plane of the congruence and any line in this plane parallel to G is a member of the congruence.

We can now parameterize the lines in the plane associated with G that are members of the fixed congruence in terms of λ, where λ is the distance of the line from G. Given,

$$G = G_1 = \left[\begin{array}{c} \mathbf{g} \\ \mathbf{g}_1^0 \end{array} \right] \tag{3}$$

and,

$$G_2 = \left[\begin{array}{c} \mathbf{g} \\ \mathbf{g}_2^0 \end{array} \right] \tag{4}$$

the lines L_G that lie in the plane defined by G_1 and G_2 and are parallel to G may be expressed as,

$$L_G = \left[\begin{array}{c} \mathbf{g} \\ (\mathbf{p}_g + \lambda\mathbf{n}) \times \mathbf{g} \end{array} \right] \tag{5}$$

where,

$$\mathbf{n} = \frac{\mathbf{g} \times (\mathbf{g}_2^0 - \mathbf{g}_1^0)}{\|\mathbf{g} \times (\mathbf{g}_2^0 - \mathbf{g}_1^0)\|} \tag{6}$$

and,

$$\mathbf{p}_g = \mathbf{g} \times \mathbf{g}_1^0 \tag{7}$$

Note that $\mathbf{n}$ is the unit vector in the direction of the common normal to the lines G_1 and G_2, that $\mathbf{p}_g$ is a point on G, and selecting $\lambda = 0$ in Eq. 5 yields the line G. In Eq. 5 we have the two dimensional set of lines of the fixed congruence associated with four spatial positions parameterized by the angle θ of the input crank of the parameterizing $4C$ mechanism, which selects a plane of the congruence, and a distance parameter λ which selects a line in that plane.

4 The Moving Congruence

The moving congruence is the two dimensional set of lines that define the moving lines of the CC dyads

that are compatible with four spatial positions of a rigid body. We obtain a parameterized representation of the moving congruence by inverting the relationship between the fixed and moving coordinate frames and proceeding in an analogous manner to the generation of the fixed congruence. The inverted positions yield the relative screw axes $S'_{12}, S'_{13}, S'_{14}, S'_{23}, S'_{24}$. We then form a complementary screw quadrilateral, its corresponding parameterizing $4C$ mechanism, and solve the spatial triangle for a given choice of θ to obtain the lines $H = H_1$ and H_2 which define a plane of the moving congruence. Proceeding as we did in the generation of the fixed congruence we obtain the lines of the moving congruence associated with the parameter θ,

$$L_H = \begin{bmatrix} \mathbf{h} \\ (\mathbf{p}_h + \mu\mathbf{n}) \times \mathbf{h} \end{bmatrix} \qquad (8)$$

where,

$$\mathbf{n} = \frac{\mathbf{h} \times (\mathbf{h}_2^0 - \mathbf{h}_1^0)}{\|\mathbf{h} \times (\mathbf{h}_2^0 - \mathbf{h}_1^0)\|} \qquad (9)$$

and,

$$\mathbf{p}_h = \mathbf{h} \times \mathbf{h}_1^0 \qquad (10)$$

Again we note that $\mathbf{p}_h$ is a point on H and that selecting $\mu = 0$ in Eq. 8 yields the line H. The result, Eq. 8, is a two dimensional set of lines, given with respect to the moving frame, associated with four spatial positions that are parameterized by the angle θ of the input crank of the parameterizing $4C$ mechanism, which selects a plane of the moving congruence, and a distance parameter μ which selects a line in that plane.

There is a one-to-one correspondence between lines of the fixed congruence and lines of the moving congruence. That is to say, selecting a line from the fixed congruence as the fixed axis of a CC dyad uniquely determines the moving axis, and vice versa, see Roth (1967a). Hence, selecting a fixed and moving line from the congruences to specify a CC dyad involves two free parameters, θ, and either λ or μ. Therefore, to uniquely determine a $4C$ mechanism from the congruences requires the selection of four free parameters; θ_1, λ_1 or μ_1 which define one dyad, and θ_2, λ_2 or μ_2 which define the second dyad. In the next section we discuss how to obtain the unknown line of a spatial CC dyad corresponding to a choice of θ and either λ or μ.

5 Completing the Dyad

Having selected a line from either the fixed or moving congruences it is now desired to determine the corresponding axis of the CC dyad which guides a body through the four spatial positions. Roth (1967a) shows that for three spatial positions of a rigid body there is a one-to-one correspondence between the moving and fixed axes of a CC dyad. Therefore, we employ the dyadic dimensional synthesis techniques of Tsai and Roth (1973) for three spatial positions to obtain the unknown corresponding line, see Larochelle (1994). We now present a method of determining the unknown fixed line of a dyad once θ and μ have been selected. Note that the procedure may be inverted to obtain the moving line given a choice of θ and λ.

For a given choice of θ and μ from Eq. 8 we have the line L_H. Since there is one-to-one correspondence between the moving and fixed axes of a CC dyad for three spatial positions we select any three of the four prescribed positions and solve for the unknown fixed axis, L_G. Moreover, because the moving and fixed congruences are both parameterized by the angle θ, and having computed the fixed congruence, we know the direction of the line L_G, $\mathbf{g}$. Hence, we need only to determine the moment of the line L_G which locates the line in the plane of the fixed congruence associated with the parameter θ. We write the crank constraint equation of the CC dyad for each of the three positions and arrive at the following system of equations which determine the moment of the unknown fixed line,

$$[P]\mathbf{g}_0 = \mathbf{b} \qquad (11)$$

where,

$$[P] = \begin{bmatrix} (\mathbf{l}_2 - \mathbf{l}_1)^T \\ (\mathbf{l}_3 - \mathbf{l}_1)^T \\ \mathbf{g}^T \end{bmatrix} \qquad (12)$$

and,

$$\mathbf{b} = \begin{bmatrix} -(\mathbf{l}_2^0 - \mathbf{l}_1^0) \cdot \mathbf{g} \\ -(\mathbf{l}_3^0 - \mathbf{l}_1^0) \cdot \mathbf{g} \\ 0 \end{bmatrix} \qquad (13)$$

and finally,

$$L_i = \begin{bmatrix} \mathbf{l}_i \\ \mathbf{l}_i^0 \end{bmatrix} \qquad (14)$$

are the coordinates of the moving line L_H in the ith position. Eq. 11 can be solved to obtain the unknown

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moment g_0. Thereby determining the fixed line L_G corresponding to a choice of moving line L_H as,

$$L_G = \begin{bmatrix} \mathbf{g} \\ \mathbf{g}_0 \end{bmatrix} \qquad (15)$$

6 Numerical Considerations

Bottema and Roth (1979) have shown that the congruences associated with four spatial positions are (9,3) congruences; nine lines(either real or imaginary) of the congruence pass through a general point and three lines(either real or imaginary) of the congruence lie in a general plane. Moreover, they have shown that at least one real line of the congruence passes through a general point. Theoretically, there are an infinite number of valid angles θ of the parameterizing linkage which yield an infinite number of planes of the congruence. Numerically, we can not generate the complete congruence. In our formulation we generate a plane of the congruence for each value of the parameter θ. Hence, by selecting a finite set of values of θ we do not generate the complete fixed and moving congruences and we do not know whether or not the planes we have generated will pass though a general point in space. We now present procedures for determining the line in a congruence nearest to some arbitrary point in space and for determining the lines in a congruence closest to some arbitrary direction.

6.1 Determining the Line Nearest to a Desired Point

We have shown that in order to uniquely determine a $4C$ mechanism from the congruences four free parameters must be specified, two for each CC dyad. In selecting a line from the congruences to define a CC dyad it may be advantageous to seek a line which passes nearest to some desired point. We now present a procedure for determining these parameters such that one of the lines of each of the dyads passes through, or comes as close as possible to, a general point $\mathbf{p}$.

First, we determine the normal distance h from a plane π of the congruence to the desired point $\mathbf{p}$. Second, we determine h for each plane of the congruence and identify the plane π_{min} which minimizes h. Finally, we compute the line L_{min} in π_{min} which is closest to the desired point $\mathbf{p}$.

Let us define the direction and moment vectors of the lines that define π as,

$$L_1 = \begin{bmatrix} \mathbf{l}_1 \\ \mathbf{l}_1^0 \end{bmatrix} \qquad (16)$$

and,

$$L_2 = \begin{bmatrix} \mathbf{l}_2 \\ \mathbf{l}_2^0 \end{bmatrix} \qquad (17)$$

recall that $\mathbf{l}_1 = \mathbf{l}_2$. We chose to represent the plane π by the implicit equation,

$$(\mathbf{p}_\pi - \mathbf{p}_0) \cdot \mathbf{n} = 0 \qquad (18)$$

where $\mathbf{n}$ is the normal vector to the plane, $\mathbf{p}_0$ is a given point in the plane, and $\mathbf{p}_\pi$ is a general point in the plane. From L_1 and L_2 we determine $\mathbf{n}$ and $\mathbf{p}_0$ as,

$$\mathbf{n} = \frac{(\mathbf{p}_2 - \mathbf{p}_1) \times \mathbf{l}_1}{\|(\mathbf{p}_2 - \mathbf{p}_1) \times \mathbf{l}_1\|} = \frac{(\mathbf{p}_2 - \mathbf{p}_1) \times \mathbf{l}_2}{\|(\mathbf{p}_2 - \mathbf{p}_1) \times \mathbf{l}_2\|} \qquad (19)$$

$$\mathbf{p}_0 = \mathbf{p}_1 = \mathbf{l}_1 \times \mathbf{l}_1^0 \text{ or } \mathbf{p}_0 = \mathbf{p}_2 = \mathbf{l}_2 \times \mathbf{l}_2^0 \qquad (20)$$

The normal distance d from the origin to the plane π is,

$$d = \mathbf{p}_0 \cdot \mathbf{n} \qquad (21)$$

We can now express $\mathbf{p}_0$ in terms of d,

$$\mathbf{p}_0 = d\mathbf{n} \qquad (22)$$

and substitute into Eq. 18 to yield,

$$\mathbf{p}_\pi \cdot \mathbf{n} = d \qquad (23)$$

Eq. 23 yields an efficient method of determining if a general point $\mathbf{p}$ lies in the plane π. If $\mathbf{p} \cdot \mathbf{n} = d \pm \epsilon$, where epsilon is some small tolerance value, then $\mathbf{p}$ is said to lie in the plane π. If $\mathbf{p} \cdot \mathbf{n} > d + \epsilon$, then $\mathbf{p}$ is said to lie in the right half-space defined by π. Similarly, if $\mathbf{p} \cdot \mathbf{n} < d - \epsilon$, then $\mathbf{p}$ is said to lie in the left half-space. Finally, the normal distance h from the plane π of the congruence, defined by L_1 and L_2, to the desired point $\mathbf{p}$ is given by,

$$h = |\mathbf{p} \cdot \mathbf{n} - d| \qquad (24)$$

The next step is to compute h for each plane of the congruence and identify the plane π_{min} which is closest to the desired point $\mathbf{p}$. Finally, we determine the line L_{min} of the congruence in π_{min} which is closest to the desired point $\mathbf{p}$. Let us refer to the lines which define π_{min} as,

$$L_{1-min} = \begin{bmatrix} \mathbf{l}_{min} \\ \mathbf{l}_{1-min}^0 \end{bmatrix} \qquad (25)$$

and,

$$L_{2-min} = \begin{bmatrix} \mathbf{l}_{min} \\ \mathbf{l}^0_{2-min} \end{bmatrix} \qquad (26)$$

Since we know the direction $\mathbf{l}_{min}$ of the lines in the plane π_{min} that are in the congruence we need only determine the moment of the unknown line L_{min}. We proceed by finding the point $\mathbf{p}_{min}$ in π_{min} which is closest to the desired point $\mathbf{p}$.

$$\mathbf{p}_{min} = \mathbf{p} - (\mathbf{p} \cdot \mathbf{n} - d)\mathbf{n} \qquad (27)$$

The unknown moment is then given by,

$$\mathbf{p}_{min} \times \mathbf{l}_{min} \qquad (28)$$

and the line L_{min} of the congruence that passes nearest to the desired point $\mathbf{p}$ is,

$$L_{min} = \begin{bmatrix} \mathbf{l}_{min} \\ \mathbf{p}_{min} \times \mathbf{l}_{min} \end{bmatrix} \qquad (29)$$

6.2 Determining the Line Nearest to a Desired Direction

In selecting a line from the congruences to define a CC dyad it may be advantageous to seek a line whose direction is nearest some desired direction. We now present a procedure for selecting such a line from a congruence.

Since every line of the congruence corresponding to a value of the parameter θ is parallel, we need only determine the plane of the congruence in which the lines are nearest to the desired direction $\mathbf{s}$. Any line in such a plane would suffice. First, we measure the error e in direction associated with a plane π of the congruence, corresponding to a value of θ as,

$$e = 1 - |\mathbf{l}_1 \cdot \mathbf{s}| = 1 - |\mathbf{l}_2 \cdot \mathbf{s}| \qquad (30)$$

where $\mathbf{l}_1 = \mathbf{l}_2$ are the direction of the lines L_1 and L_2 which define the plane π. We compute e for each plane of the congruence and determine the plane π_{min} which minimizes e. Any line in the plane π_{min} which lies in the congruence is nearest the desired direction and may be used to define a line of a CC dyad which guides a body through the four prescribed positions.

7 Case Study

In this section we present an example of the design of a spatial $4C$ mechanism for four position rigid body guidance. The goal is to move a pallet off of a flexible

Pos.	X	Y	Z	Long.	Lat.	Roll
1	0.00	0.00	0.00	00.0	00.0	00.0
2	0.00	1.00	0.25	15.0	15.0	00.0
3	1.00	2.00	0.50	45.0	60.0	00.0
4	2.00	3.00	1.0	45.0	80.0	00.0

Table 1: The 4 Prescribed Positions

assembly line into a convenient position to perform an assembly operation on the underside of the pallet and then to return the pallet to the assembly line. A moving coordinate frame was assigned to the pallet and the $4C$ mechanism is to attach itself to the pallet at the points; $[-1\ 0.25\ 0]^T$ and $[1\ 0.25\ 0]^T$. It is at these two points that holes are to be drilled in the pallet. These holes will serve as journal bearings for the moving C joints of the $4C$ mechanism. This application was suggested by Mark Senti and his associates at GSMA Systems Inc, Melbourne, FL. The four desired positions are prescribed by the (X, Y, Z) coordinates of the origin of the moving frame and the $(Longitude, Latitude, Roll)$ angles which describe the orientation of the moving frame with respect to the fixed reference frame, see Tbl. 1. The fixed and moving congruences, shown in Fig. 3, were generated for sequential values of θ beginning at $\theta = 0.0$ and stepping in increments of $0.1\text{rad} = 5.73\,\text{deg}$. The values of d chosen for computing the two lines for a given value of θ were $d_1 = 1.0$ and $d_2 = 2.0$.

From the computed congruences we seek a $4C$ mechanism with a driving crank which has a moving line that passes through the point $\mathbf{p}_{dvg} = [-1\ 0.25\ 0]^T$ and a driven crank with a moving line that passes through the point $\mathbf{p}_{dvn} = [1\ 0.25\ 0]^T$; both points are given with respect to the moving frame. First, we determine the driving dyad. For each plane of the moving congruence the distance h from the plane to the point $\mathbf{p}_{dvg}$ was computed. The plane corresponding to $\theta = 183.35$ was found to be nearest the point $\mathbf{p}_{dvg}$, $h_{min} = 0.032135$, and the line in this plane nearest to $\mathbf{p}_{dvg}$ is,

$$H_{dvg} = \begin{bmatrix} 0.79989 \\ -0.59803 \\ 0.05039 \\ 0.01387 \\ 0.54772 \\ 0.42987 \end{bmatrix} \qquad (31)$$

The point on this line nearest to $\mathbf{p}_{dvg}$ is $\mathbf{p}_{min} = [-1.01924\ 0.22462\ 0.00426]^T$. The corresponding fixed line of the CC dyad, whose moment was found
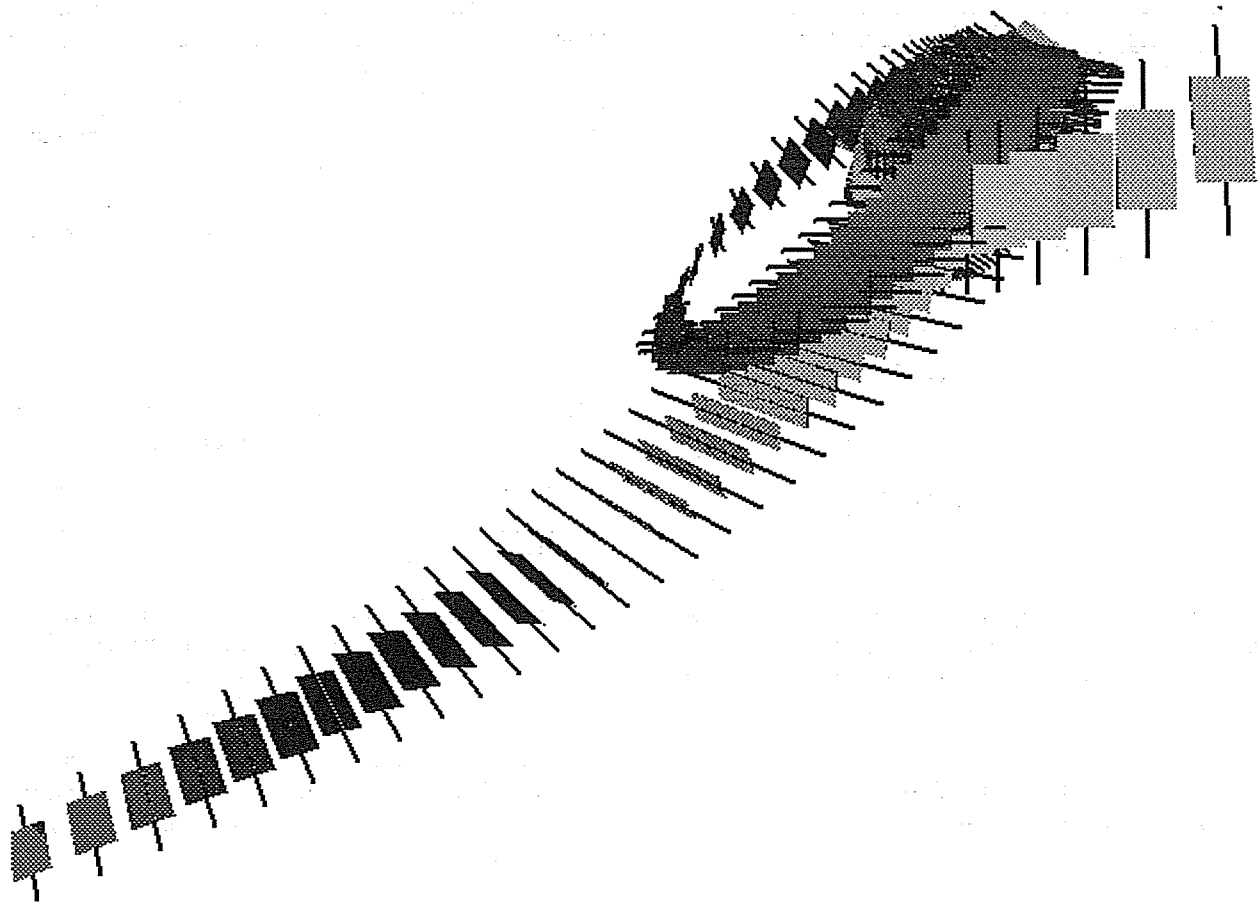
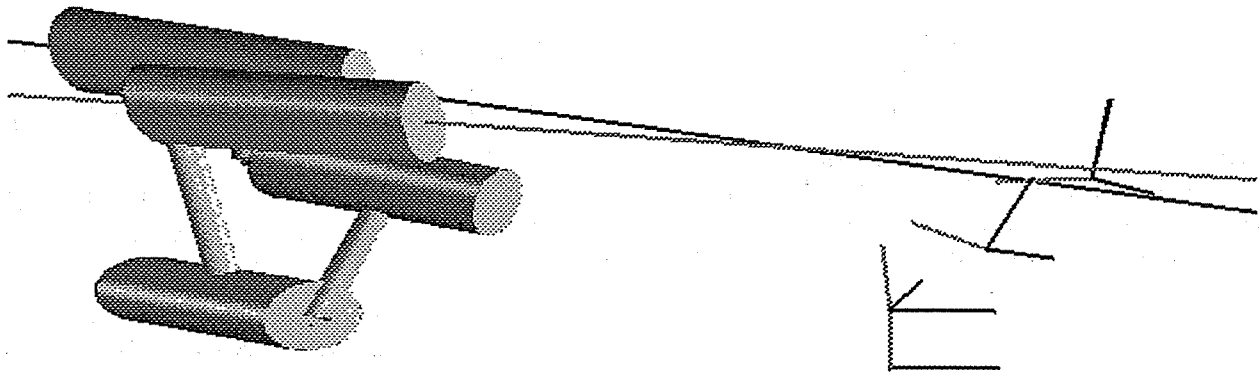Figure 3: The Fixed and Moving Congruences

Figure 4: The $4C$ Mechanism

Link	Length (deg, distance)
DRIVING	(25.81, 2.718)
COUPLER	(7.73, 0.668)
DRIVEN	(−165.41, 1.219)
FIXED	(−161.19, 2.112)

Table 2: The Desired $4C$ Mechanism

using three position synthesis, is,

$$G_{dvg} = \begin{bmatrix} 0.93653 \\ -0.23042 \\ 0.26423 \\ 0.59294 \\ 3.00646 \\ 0.52020 \end{bmatrix} \tag{32}$$

We obtain the driven dyad in an analogous manner. The plane corresponding to $\theta = 246.37$ was found to be nearest the point p_{dvn}; $h_{min} = 0.019724$, and the line in this plane of the moving congruence nearest to p_{dvn} is,

$$H_{dvn} = \begin{bmatrix} 0.86004 \\ -0.50927 - 0.03126 \\ -0.01677 \\ 0.01674 \\ -0.73418 \end{bmatrix} \tag{33}$$

The point on this line nearest to p_{dvn} is $p_{min} = [1.00459\ 0.25880\ -0.01704]^T$. The corresponding fixed line of the driven dyad is,

$$G_{dvn} = \begin{bmatrix} -0.95907 \\ 0.27706 \\ 0.05847 \\ -0.25088 \\ -1.15251 \\ 1.34609 \end{bmatrix} \tag{34}$$

Finally, we show the resulting $4C$ mechanism in Fig. 4 and list its link lengths in Tbl. 2. In Fig. 4 the mechanism is shown with the moving body in position 4 and the lines which attach the moving body to the mechanism are also shown.

8   Conclusion

In this paper we have presented a procedure for determining the fixed and moving congruences associated with four finitely separated spatial positions. Moreover, algorithms for selecting lines from the congruences based upon two design criteria: selecting lines which pass through, or near, desired points and lines which are nearest some desired direction, were derived. The result is a design procedure for performing the kinematic dimensional synthesis of spatial $4C$ mechanisms for four position rigid body guidance. The design process was illustrated in a detailed example. It is hoped that this procedure will facilitate the creation of computer aided design software for spatial $4C$ mechanisms.

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